

Physical aspects of the field-theoretical description of two-dimensional ideal fluids

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December 31, 2009

Abstract

The two-dimensional ideal (Euler) fluids can be described by the classical fields of streamfunction, velocity and vorticity and, in an equivalent manner, by a model of discrete point-like vortices interacting in plane by a self-generated long-range potential. This latter model can be formalized, in the continuum limit, as a field theory of scalar matter in interaction with a gauge field, in the $su(2)$ algebra. This description has already offered the analytical derivation of the *sinh*-Poisson equation, which was known to govern the stationary coherent structures reached by the Euler fluid at relaxation. In order this formalism to become a familiar theoretical instrument it is necessary to have a better understanding of the physical meaning of the variables and of the operations used by the field theory. Several problems will be investigated below in this respect.

1 Introduction

The two-dimensional incompressible ideal fluid is governed by the Euler equation

$$\frac{d\boldsymbol{\omega}}{dt} = 0 \quad (1)$$

where $\boldsymbol{\omega}$ is the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{v}$, a vector perpendicular on the plane where the flow with the velocity \mathbf{v} is contained. It is useful to define the streamfunction ψ from which the velocity field is derived $\mathbf{v} = \hat{\mathbf{e}}_z \times \nabla \psi$ and

the vorticity is $\omega = \hat{\mathbf{e}}_z \Delta \psi$. Most of the results in the theory of the ideal 2D Euler fluid have been obtained using these three quantities, which have clear physical meaning and are measurable experimentally.

On the other hand the striking result that the asymptotic stationary states obtained at relaxation exhibit a strong coherency of the flow cannot be easily described in the framework defined by $(\psi, \mathbf{v}, \omega)$. The stationary flows from Eq.(1) obey the equation $[(-\nabla \psi \times \hat{\mathbf{e}}_z) \cdot \nabla] \nabla^2 \psi = 0$ which has a very large space of possible solutions. Clearly the selection of the final states is dictated by an additional constraint that may have the form of a functional extremal condition. Finding a functional defined on the space of flow configuration, as for example a density of a Lagrangian and an action functional, has not been possible in the traditional approach.

A model (Hamilton, Kirchhoff, Onsager) which is equivalent with the 2D Euler fluid consists of a set of discrete point-like vortices interacting in plane by a potential which is created by themselves and has long range, *i.e.* it is Coulombian

$$\begin{aligned} \frac{dx_i}{dt} &= -\frac{\partial}{\partial y} \sum_{j \neq i}^N \omega_0 \ln \left(\frac{|\mathbf{x} - \mathbf{x}_j|}{L} \right) \\ \frac{dy_i}{dt} &= \frac{\partial}{\partial x} \sum_{j \neq i}^N \omega_0 \ln \left(\frac{|\mathbf{x} - \mathbf{x}_j|}{L} \right) \end{aligned} \quad (2)$$

The following observation will have an important consequence later in the theory. In any further development it is not sufficient to only work with the system (2). The system (2) as it is still needs to specify which are the *objects* whose positions (x_i, y_i) evolve in plane.

The elementary objects can be *charges*. For large N the system can be treated as a statistical ensemble. In particular it has been shown that it has negative temperature (S.F. Edwards and J. B. Taylor) which suggests an intrinsic tendency to self-organization into large structures.

On the other hand the discrete elementary objects can be *vortices* as actually is requested by the equivalence with the Euler fluid, where the natural logarithm appears as the propagator inverting the equation relating the vorticity and the streamfunction, $\Delta \psi = \omega$ (Kraichnan, Montgomery). Again in a statistical approach, the stationary states corresponding to the maximum entropy under the conservation of the mass, momentum and energy have been found to be described by the differential equation *sinh*-Poisson

$$\Delta \psi + \sinh \psi = 0 \quad (3)$$

where ψ is the streamfunction.

The problem is how to differentiate between the system consisting of Eqs.(2) *plus* the information that the objects are charges and the system consisting of Eqs.(2) *plus* the information that the objects are vortices. S.F. Edwards and J. B. Taylor observe that the two-dimensional plasma (*i.e.* charges) and the vortex fluid (*i.e.* vortices) are described by the same system of equations. The system of *charges* and the system of *vortices* are formally identical if we replace the charge by the circulation. However for statistical considerations it is not sufficient. The field theory clearly shows this, obtaining the distinct results that the system of charges is described by the Liouville equation and the system of vortices by the *sinh*-Poisson equation. The field theory describes the charges by an Abelian model and a vortex fluid by a non-Abelian one.

The *sinh*-Poisson equation has been confirmed by numerical simulations of relaxation of Euler fluid from states of turbulence to the coherent and quasi-stationary asymptotic states (Montgomery).

In consequence of the discussion above we note that the derivation of the *sinh*-Poisson equation from purely statistical consideration still requires some care. In addition the statistical approach could not be extended to other type of fluids like the 2D atmosphere or the plasma in strong magnetic field. However the fact that the streamfunction ψ verifies the *sinh*-Poisson equation has been confirmed after careful numerical verification. Any new theory of the Euler fluid will have to be confronted to this challenge, to derive the *sinh*-Poisson equation.

The field theoretical (FT) model for the Euler fluid (Spineanu and Vlad, 2003) is able to provide a purely analytical derivation of the *sinh*-Poisson equation. Moreover it can be extended to the more complicated problem of the 2D plasma in strong transversal magnetic field and 2D planetary atmosphere. The FT models for fluid, plasma, atmosphere have clearly shown that they are able to derive new results which are inaccessible to the traditional approaches based on $(\psi, \mathbf{v}, \boldsymbol{\omega})$. However these FT models would be more easily adopted as an instrument of theoretical investigation if there would be a physical understanding of the meaning of the field theoretical concepts, operations, etc. in terms of the more familiar $(\psi, \mathbf{v}, \boldsymbol{\omega})$.

2 The field theoretical representation of the elementary vortex as a global string

The definition of the elementary object in the discrete model of point-like vortices consists of two characteristics:

1. the elementary object in plane is strictly point-like, there is no spatial extension. We can represent it however as a line extending in the z direction, *i.e.* perpendicular on the plane
2. the elementary object carry a vorticity “content” which, although it is not a scalar charge, it is not introduced as the result of a fluid rotating around the vortex

For the reason resulting from the second characteristic, the convenient representation of the elementary vortex in field theory is the global string. This is a time-independent solution of the equation of motion of a spontaneously broken global $U(1)$ Higgs model (Davies and Shellard). By this representation we dispose of a field theory for the elementary vortex as a Nambu-Goldstone boson $\alpha(x)$ with the complex scalar field

$$\hat{\phi} = \eta \exp(i\alpha) \tag{4}$$

where η is the vacuum value of $\hat{\phi}$. Later this model will prove to be essential in the investigation of the fermionic zero-modes propagating along the string and strictly confined to it. The axial anomaly related to these modes interacting with a gauge field will allow us to get an understanding of the vorticity concentration, a process of high importance in the tropical cyclone and tornadoes.

For the global string there is no Magnus force acting on the string when it is moving with velocity \mathbf{v} in the fluid. This is not a problem since the laws of motion of the point-like objects is given by the purely kinematic equations of motion and the motion resulting from these equations is not regarded as a physical interaction with a fluid environment.

However we can not be satisfied with the definition of the elementary vortex as a global string *plus* the statement that it carries a fixed vorticity. As shown by Davies and Shellard the global string gets a fixed vorticity when it is assumed that there is interaction between the string and a background field which breaks the Lorentz invariance and is equivalent with inducing rotation of the global string around its axis, or a linear time variation of the bosonic field $\alpha = \theta + \text{const} \times t$ with θ the azimuthal angle. Taking the energy

density of the pure background field p the fixed vorticity carried by a global string in such background is

$$\omega_0 = (4\pi\eta/\sqrt{p}) \hat{\mathbf{e}}_z \quad (5)$$

This is a mechanism by which we can attach a fixed vorticity to the global string but it is not less arbitrary than defining the elementary object as carrying a fixed vorticity ω_0 .

3 The physical meaning of the concepts involved in the field theoretical model of the Euler fluid

3.1 Review of field theoretical model for the Euler fluid

Jackiw and Pi have developed a field-theoretical for the continuum limit of a system of charges in plane. The equations of motion can be derived from the density of a Lagrangian and the theory leads to a differential equation describing the asymptotic stationary states of the system. This is the Liouville equation.

We have found that the continuum limit of the discrete model of *vortices* is given by a field theory with the Lagrangian that is characterized by: non-relativistic (Schrodinger), Chern-Simons, 4th order scalar field self-interaction. The Lagrangian is

$$\begin{aligned} L = & -\kappa\varepsilon^{\mu\nu\rho}\text{tr}\left((\partial_\mu A_\nu)A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right) \\ & +i\text{tr}\left(\phi^\dagger(D_0\phi)\right) - \frac{1}{2}\text{tr}\left((D_k\phi)^\dagger(D^k\phi)\right) \\ & +\frac{1}{4\kappa}\text{tr}\left([\phi^\dagger,\phi]^2\right) \end{aligned} \quad (6)$$

where $\phi, \phi^\dagger, A_\mu, A_\mu^\dagger$ are $SU(2)$ elements. The covariant derivatives are

$$D_\mu = \partial_\mu + [A_\mu, \cdot] \quad (7)$$

with the metric $g^{00} = -1, g^{11} = g^{22} = 1$. The equations of motion are

$$\begin{aligned} iD_0\phi &= -\frac{1}{2m}D_k D^k\phi - \frac{1}{2m\kappa}[[\phi, \phi^\dagger], \phi] \\ -\kappa\varepsilon^{\mu\nu\rho}F_{\nu\rho} &= iJ^\mu \end{aligned} \quad (8)$$

where

$$\begin{aligned} J^0 &= [\phi^\dagger, \phi] \\ J^k &= -\frac{i}{2m} \left([\phi^\dagger, (D^k \phi)] - [(D^k \phi)^\dagger, \phi] \right) \end{aligned} \quad (9)$$

which is *covariantly conserved* $D_\mu J^\mu = 0$. The action functional calculated with this Lagrangian has the fundamental property that can be written in the Bogomolnyi form, *i.e.* as a sum of squares, which clearly identifies the absolute extrema in the space of states by simply taking these square terms to zero. The energy is

$$E = \frac{1}{2} \text{tr} \left((D_k \phi)^\dagger (D^k \phi) \right) - \frac{1}{2\kappa} \text{tr} \left([\phi^\dagger, \phi]^2 \right) \quad (10)$$

The Gauss law is the zero component of the second equation of motion

$$-2\kappa F_{12} = iJ^0 \quad (11)$$

Introducing the notations $D_\pm = D_x \pm iD_y$ and similarly for other variables, the first term in Eq.(10) is

$$\text{tr} \left((D_k \phi)^\dagger (D^k \phi) \right) = \text{tr} \left((D_- \phi)^\dagger (D_- \phi) \right) - i \text{tr} \left(\phi^\dagger [F_{12}, \phi] \right) \quad (12)$$

Replacing in the expression of the energy, the last term, coming from the *potential* energy or the scalar field self-interaction is canceled and we obtain

$$E = \frac{1}{2} \text{tr} \left((D_- \phi)^\dagger (D_- \phi) \right) \quad (13)$$

which leads to the equation for the states which correspond to the lowest energy

$$D_- \phi = 0 \quad (14)$$

Then the equations of motion are replaced by

$$\begin{aligned} D_- \phi &= 0 \\ \partial_+ A_- - \partial_- A_+ + [A_+, A_-] &= \frac{1}{\kappa} [\phi, \phi^\dagger] \end{aligned} \quad (15)$$

The solutions are stationary flow configurations obeying the set (15) of two first order partial differential equations from which, under a reasonable algebraic *ansatz* one can derive the *sinh*-Poisson equation, a unique differential

equation for the streamfunction. The algebraic ansatz uses the generator of the Cartan sub-algebra and the two ladder generators

$$\begin{aligned}\phi &= \phi_1 E_+ + \phi_2 E_- \\ \phi^\dagger &= \phi_1^* E_- + \phi_2^* E_+\end{aligned}\tag{16}$$

$$\begin{aligned}A_+ &= A_x + iA_y = aH \\ A_- &= A_x - iA_y = -a^*H\end{aligned}\tag{17}$$

from which the first SD equation leads to

$$a = \frac{\partial}{\partial z^*} \ln(\phi_1^*)\tag{18}$$

$$a^* = \frac{\partial}{\partial z} \ln(\phi_1)\tag{19}$$

and

$$\begin{aligned}\text{tr}(\phi\phi^\dagger) &= \rho_1 + \rho_2 \\ [\phi, \phi^\dagger] &= (\rho_1 - \rho_2)H\end{aligned}\tag{20}$$

with the notation

$$\begin{aligned}\rho_1 &\equiv |\phi_1|^2 \\ \rho_2 &\equiv |\phi_2|^2\end{aligned}\tag{21}$$

Using the algebraic ansatz in Eq.(15) we obtain the *sinh*-Poisson equation.

The particular property of the system allowing the Bogomolnyi form of the action is called Self-Duality. It is worth to mention that the field theoretical formalism for the Euler fluid has been able to show that the asymptotic coherent structures of the Euler fluid belong to the same family as all the other structures known: the solitons, the instantons, all are solutions of equations derived at Self-Duality. It would be desirable to have the physical meaning of this concept in terms of classical fluid variables, *i.e.* $(\psi, \mathbf{v}, \boldsymbol{\omega})$.

3.2 Connections derived from the fermionic nature of the elementary point-like vortex

The elementary object of the discrete model is a point-like vortex. The magnitude is the same for all elementary vortices, say ω_0 . It is not admitted

to work with multiple vortices as single objects, for example consisting of two elementary objects superposed into a unique vortex of magnitude $2\omega_0$. This can be expressed by saying that two elementary vortices cannot be in the same point. There are only two possible orientations, given by $\pm\omega_0$. We conclude that there are similarities between the elementary point-like vortex and a spin-1/2 object. Further this means that if one looks for a field theoretical formulation of the model of point-like vortices the most natural way would involve fermionic fields.

On the other hand, Jackiw and Pi have shown that a system of point electric charges moving in plane according to the same Eqs.(2) can be described by a classical Abelian field theory of a bosonic scalar and gauge fields with Chern-Simons term and with φ^4 scalar field self-interaction. This model has self-dual states that are described by the Liouville equation, $\Delta\psi + \exp(\psi) = 0$. Since however the Euler fluid is described by point-like *vortices* instead of *electric charges*, the field theory must be re-formulated to reflect the spinorial nature of the elementary objects. Since the spinors are the lowest representations of the Lorentz group whose complex covering is $SL(2, \mathbf{C})$ we expect to represent the spinorial nature of the elementary vortices by taking all bosonic variables of the Jackiw-Pi model as elements of the $sl(2, \mathbf{C})$ algebra. This is the model of Eq.(6) from which the *sinh*-Poisson equation has been derived. This places the non-Abelian, bosonic scalar field model, with Chern-Simons term for the gauge field and scalar potential nonlinearity of order four Eq.(6) as the main framework where we should look for more conventional physical significance. However we also note the possibility that neighboring “fermionic” models can be useful. The connection should be realised via bosonization procedures, where it is possible.

There are several two-dimensional models that present similarities with the model of point-like vortices and that have received considerable attention, with various purposes: the Thirring model, the Schwinger model, the Nambu-Jona-Lasinio model, etc.

We note that the scalar-field self-interaction of order four, which is in Non-Abelian form

$$\frac{1}{2}\text{tr}\left([\phi^\dagger, \phi]^2\right) \quad (22)$$

or, in Abelian case

$$\frac{1}{2}(\phi^*\phi)^2 \quad (23)$$

is the same as the fourth order term in the expansion

$$\cos \phi - 1 \approx -\frac{1}{2}(\phi^*\phi) + \frac{1}{24}(\phi^*\phi)^2 \quad (24)$$

The term $(\phi^*\phi)$ should be considered separately together with the mass term. If the fourth order scalar-field non-linearity $\sim (\phi^*\phi)^2$ comes from $\cos(\phi) - 1$ then the nonlinearity is the same as in the *sine-Gordon* model. Then the fermionic model to which we should look is the Thirring model with the Lagrangian

$$\begin{aligned}\mathcal{L}_{Th} = & -\bar{\psi} \left(i\partial_\mu \right) \psi \\ & -\frac{1}{2} (\bar{\psi}\gamma^\mu\psi) (\bar{\psi}\gamma_\mu\psi)\end{aligned}\tag{25}$$

which is characterized by a current-current (JJ) interaction. Writting the nonlinearity as

$$(\bar{\psi}\gamma^\mu\psi) (\bar{\psi}\gamma_\mu\psi) = \frac{1}{2} \left[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2 \right]\tag{26}$$

We propose the following identification

$$\begin{aligned}(\bar{\psi}\psi)^2 &= (\rho_1 + \rho_2)^2 \\ (\bar{\psi}\gamma_5\psi)^2 &= (\rho_1 - \rho_2)^2\end{aligned}\tag{27}$$

such that

$$\begin{aligned}& \frac{1}{2} \left[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2 \right] \\ &= \frac{1}{2} \left[(\rho_1 + \rho_2)^2 - (\rho_1 - \rho_2)^2 \right] \\ &= 2\rho_1\rho_2\end{aligned}\tag{28}$$

By this identification JJ is connected with the product

$$\rho_1\rho_2\tag{29}$$

which plays a major role in the field theory for Euler. The extremum of the action for the Thirring model is

$$JJ = \text{const}\tag{30}$$

and this means, after normalizations,

$$\rho_1\rho_2 = 1\tag{31}$$

This equation plays a major role in the field-theoretical derivation of *sinh*-Poisson equation for Euler and demands an interpretation in physical

terms. The identification proposed above can open the way to a physical interpretation.

The two components are

$$\begin{aligned} (\bar{\psi}\psi) &= \psi^\dagger \gamma^0 \psi = \psi^\dagger \sigma^3 \psi \quad \text{the density of spin} \\ (\bar{\psi}\gamma_5\psi) &\quad \text{chiral current} \end{aligned} \quad (32)$$

We recall (**Coleman, Faber and Ivanov**) that the two models: **sine-Gordon** (bosonic) and **Thirring** (fermionic) are equivalent if $4\pi/\beta^2 = 1 + g/\pi$ and the fermionic field $\psi(x)$ respectively the bosonic field $\theta(x)$ satisfy the Abelian bosonization relation

$$m\bar{\psi}(x) \left(\frac{1 \mp \gamma^5}{2} \right) \psi(x) = -\frac{\alpha}{\beta^2} \exp [\pm i\beta\theta(x)] \quad (33)$$

(here the constants g, β, α, m , are constants). **Witten** uses the notations

$$\bar{\psi}(x) \left(\frac{1 \pm \gamma^5}{2} \right) \psi(x) \equiv \mathcal{O}_\pm \quad (34)$$

which are called *chiral densities*. According to the identification our $\rho_{1,2}$

$$\rho_1, \rho_2 \sim \mathcal{O}_\pm \quad (35)$$

the coefficients of the ladder generators. Then we should compare

$$\begin{aligned} \bar{\psi}(x) \left(\frac{1 + \gamma^5}{2} \right) \psi(x) &\iff \phi_1^* \phi_1 \quad (\text{from } \phi_1 E_+) \\ \bar{\psi}(x) \left(\frac{1 - \gamma^5}{2} \right) \psi(x) &\iff \phi_2^* \phi_2 \quad (\text{from } \phi_2 E_-) \end{aligned} \quad (36)$$

We have

$$\begin{aligned} \text{tr}(\phi\phi^\dagger) &= \rho_1 + \rho_2 \\ [\phi, \phi^\dagger] &= (\rho_1 - \rho_2) H \end{aligned} \quad (37)$$

$$\begin{aligned} \rho_1 - \rho_2 &= [\phi^\dagger, \phi] / H = \text{vorticity Euler } \omega \\ \downarrow \\ \bar{\psi}(x) \gamma^5 \psi &\quad (\text{axial current}) \end{aligned} \quad (38)$$

$$\begin{aligned} \rho_1 + \rho_2 &= \text{tr}(\phi\phi^\dagger) \\ \downarrow \\ \bar{\psi}(x) \psi(x) &\quad \text{density of spin} \end{aligned} \quad (39)$$

In conclusion a possible identification is

$$\begin{aligned}\mathcal{O}_+ &= \bar{\psi} \left(\frac{1 + \gamma^5}{2} \right) \psi = \phi_1^* \phi_1 = \rho_1 = \text{density of positive-chirality modes} \\ \mathcal{O}_- &= \bar{\psi} \left(\frac{1 - \gamma^5}{2} \right) \psi = \phi_2^* \phi_2 = \rho_2 = \text{density of negative-chirality modes}\end{aligned}\quad (40)$$

$$\begin{aligned}\bar{\psi}\psi &= \rho_1 + \rho_2 = \text{tr}(\phi^\dagger \phi) \text{ density of spin (Boyanovsky)} \\ \bar{\psi}\gamma^5\psi &= \rho_1 - \rho_2 = [\phi^\dagger, \phi] / H = \omega \text{ axial current}\end{aligned}\quad (41)$$

From Eq.(33) we have

$$\begin{aligned}\bar{\psi}\psi - \bar{\psi}\gamma^5\psi &= -\frac{2\alpha}{m\beta^2} \exp(i\beta\theta) \\ \bar{\psi}\psi + \bar{\psi}\gamma^5\psi &= -\frac{2\alpha}{m\beta^2} \exp(-i\beta\theta)\end{aligned}\quad (42)$$

$$\begin{aligned}\bar{\psi}\psi &= -\frac{2\alpha}{m\beta^2} \cos(\beta\theta) \\ \bar{\psi}\gamma^5\psi &= i\frac{2\alpha}{m\beta^2} \sin(\beta\theta)\end{aligned}\quad (43)$$

$$\begin{aligned}(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma^5\psi)^2 &= \left(\frac{2\alpha}{m\beta^2} \right)^2 [(\cos(\beta\theta))^2 + (\sin(\beta\theta))^2] \\ &= \left(\frac{2\alpha}{m\beta^2} \right)^2\end{aligned}\quad (44)$$

which could correspond to the equation

$$(\rho_1 + \rho_2)^2 - (\rho_1 - \rho_2)^2 = \left(\frac{2\alpha}{m\beta^2} \right)^2 \quad (45)$$

or

$$\rho_1 \rho_2 = \frac{1}{4} \left(\frac{2\alpha}{m\beta^2} \right)^2 \quad (46)$$

The fact that none of ρ_1 or ρ_2 can be zero results from the fact that each represents a *chiral density*

$$\begin{aligned}\mathcal{O}_+ &= \rho_1 \\ \mathcal{O}_- &= \rho_2\end{aligned}\quad (47)$$

The vanishing of any of the two densities ρ_1 or ρ_2 would lead to the equation

$$\exp(i\theta) = 0 \quad (48)$$

with no solution. The fact that none of these two functions can vanish is essential for the structure of the differential equation describing the self-dual states: the *sinh*-Poisson equation contains both $\exp(+\psi)$ and $\exp(-\psi)$. If we think to the derivation of this equation, reviewed above, it leaves the impression that the value of the vorticity in a particular point is always a result of two opposite actions, and none of them can be absent: creation of vorticity in a small spatial patch by the effect of densification of point-like vortices by the action of the ladder generator E_+ followed by decrease of the local vorticity by rarefaction of elementary vortices in the same patch, realised by the second generator E_- . Actually we see that the FT operations mean the subtraction of the two chiral densities $O_+ - O_-$ such that the spin density is eliminated leaving only the chiral current, *i.e.* vorticity.

3.3 Review of the connections and comments on the interpretation

The connection goes through these steps:

- the Euler equation
- the point-like vortices
- the Jackiw-Pi model, constructed for *charged* point-like objects
- the non-relativistic Non-Abelian $SU(2)$ CS 4th; this is for point-like *vortices*.
- observation that the scalar-field part is close at low amplitudes to the sine-Gordon model with the nonlinearity $\cos \phi - 1$ expanded to second order, *i.e.* to ϕ^4 term.
- exploiting by “anti-bosonization” the connection with the Thirring *fermionic* non-linearity, JJ .
- identification of the *chiral densities* defined within the Thirring model as the amplitudes of the algebraic operations of the ladder generators in the *ansatz* for the $su(2)$ scalar field ϕ of the FT model for Euler.
- JJ constant, leading to SD states

If this identifications are confirmed then it formulates the field theoretical model in a more accessible way, preparing for physical discussions in terms of classical fluid approaches.

We need a non-Abelian field theory (FT) and TWO functions, ρ_1 and ρ_2 to describe the Euler fluid. This comes from the nature of the point-like object, which is a *vortex* is not a charge like in Jackiw-Pi. For Jackiw-Pi model an Abelian charge was sufficient and the result was the Liouville equation. The intermediate Thirring-sine-Gordon mapping (bosonization) is fully *Abelian*. Since for the Euler fluid the point-like objects carry vorticity the model must be *non-Abelian* and the final equation at self-duality is *sinh*-Poisson.

Taking the physical streamfunction as Ψ , at self-duality in FT we have

$$\begin{aligned} (\bar{\psi}\gamma_5\psi) &= \omega \\ &= \rho_1 - \rho_2 \\ &= \sinh \Psi \end{aligned} \tag{49}$$

and

$$\begin{aligned} (\bar{\psi}\psi) &= \rho_1 + \rho_2 \\ &= \cosh \Psi \end{aligned} \tag{50}$$

and

$$\begin{aligned} JJ &= (\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2 \\ &= (\cosh \Psi)^2 - (\sinh \Psi)^2 \\ &= 1 \end{aligned} \tag{51}$$

which means that SD is equivalent with constant interaction JJ .

4 Discussion of the meaning of $\rho_1\rho_2 = 1$ in the non-Abelian model

We have seen the meaning of the relation $\rho_1\rho_2 = 1$ in the case where we use the fermionic counterpart of the theory. Now we return to the bosonic, non-Abelian model.

For the Euler fluid the condition $\rho_1\rho_2 = 1$ arises from the equality of two distinct expression for the magnetic field F_{+-} calculated first with only ϕ_1 and alternatively with only ϕ_2 . This is possible because the algebraic operation involved in the first self-duality equation

$$D_-\phi = 0 \tag{52}$$

preserves without mixing the generators of the algebra in the original places where they exist according to the algebraic *ansatz*. The commutators reproduce the same generator (with change of sign for E_-) and the equality with zero gives two equations in which the potentials A_{\pm} are expressed either by ϕ_1 or by ϕ_2 . It is now essential that the algebraic *ansatz* connects formally the two potentials $A_+ = aH$ and $A_- = -a^*H$. The operations that are performed in the equation $D_- \phi = 0$ involve commutators of A_- (from the covariant operator D_-) with the function ϕ , or

$$[H, E_{\pm}] = \pm 2E_{\pm} \quad (53)$$

It is essential that the commutators do NOT mix the algebra generators. If A_- were *not* along the Cartan generator H but it contained E_{\pm} then there would have been mixing of generators. They remain separate and the two equations derived from $D_- \phi = 0$ involve ϕ_1 and respectively ϕ_2 , separately. This looks like an equation of eigenfunctions where the generator H (which is the potential A_-) leaves invariant the ladder E_+ , as if it was its eigenfunction. This means that the potential which is the fluid physical velocity does not affect the ladder generator when it is compatible with it.

It is equally important that the function a and its complex conjugate a^* are the potentials A_{\pm} . This means physically to consider the same velocity in the transversal plane, acting to determine ϕ_1 and respectively ϕ_2 . The two functions A_+ and A_- represent, in this algebraic *ansatz* the same physical velocity in the plane but seen from two opposite directions, from up and from down but along the same z direction. This means that ϕ_1 and ϕ_2 must be connected and finally the relationship

$$\rho_1 \rho_2 = 1 \quad (54)$$

is obtained.

We can express the magnetic field F_{+-} in two ways, using each time only *one* variable ϕ_1 or ϕ_2 .

In conclusion, we have two ways to think the Eq.(54): $JJ = \text{const}$, in every point. The other way is:

$$\Delta \ln |\phi_1|^2 + \Delta \ln |\phi_2|^2 = 0$$

coming from the fact that we can express the field F_{+-} in two different ways, using either ϕ_1 (which means increase by E_+) or ϕ_2 (using E_-).

This is the reason for $\rho_1 \rho_2 = 1$.

5 Conclusion

Less than a physical interpretation, the present discussion may be useful for further investigation of the connection between the traditional approach to the fluid physics and the field theoretical approach.

Acknowledgement 1 *This work has been supported by the Romanian Ministry of Education and Research under Ideas Exploratory Research Project No.557*

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